The Integration of $\delta'(f)$ in a Multidimensional Space

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1.0 Introduction

In the study of noise from high speed surfaces, one needs to evaluate integrals involving $\delta'(f)$ where $f(\boldsymbol{x}) = 0$ or $f(\boldsymbol{x}, t) = 0$ is a stationary or moving surface on which acoustic sources lie. For example, consider the thickness noise term of the Ffowcs williams-Hawkings (FW-H) equation when we take the time derivative explicitly:

$$\begin{aligned} \frac{\partial}{\partial t}[\rho_0 v_n \delta(f)] &= \frac{\partial}{\partial t}[\rho_0 \underset{\sim}{v}_n \delta(f)] \\ &= \rho_0 \dot{\underset{\sim}{v}}_n \delta(f) - \rho_0 \underset{\sim}{v}_n v_n |\ \ f| \delta'(f) \end{aligned} \quad \text{(EQ 1)}$$

It is assumed that the function $f(\boldsymbol{x}, t)$ is so defined that we have $|\ \ f| = 1$ on this surface. The local normal velocity of the surface is denoted v_n given by the relation $v_n = -\partial f / \partial t$ and a tilde under a symbol indicates restriction of the variable to the surface $f = 0$ [1]. This relation shows how the generalized function $\delta'(f)$ appears in wave propagation problems as the source term of the linear wave equation. We point out two features of Eq. (1). First we note that only one of the two normal derivatives multiplying $\delta'(f)$ is restricted to the surface $f = 0$. Second, the unrestricted function $|\ \ f|$ also appears as a factor of $\delta'(f)$ which can not and should not be set equal to 1.

In this paper, we will first give the interpretation of the following integral

$$I = \int \varphi(\boldsymbol{x}) \delta'(f) d\boldsymbol{x} \quad \text{(EQ 2)}$$

where we will initially assume that $|\quad f| \neq 1$ on the surface $f = 0$. We next give two more useful results by first assuming that

$$\varphi(\boldsymbol{x}) = Q(\boldsymbol{x})\left|\ \underset{\sim}{f}\right||\quad f| \qquad \text{(EQ 3)}$$

for some arbitrary C^1 function $Q(\boldsymbol{x})$ and then set $|\quad f| = 1$ on the surface $f = 0$.

2.0 Derivation of the Main Result

We use Gaussian coordinates (u^1, u^2) on the surface $f = 0$ and then extend these coordinates along local normals to the space near the surface. Next we take $u^3 = f$. We have

$$d\boldsymbol{x} = \sqrt{g_{(3)}}du^1du^2du^3 = \frac{\sqrt{g'_{(2)}}}{|\quad f|}du^1du^2du^3 \qquad \text{(EQ 4)}$$

Here $g_{(3)} = g'_{(2)}/|\quad f|^2$ is the determinant of the coefficients of the first fundamental form in the new variables. The symbol $g'_{(2)}$ stands for the determinant of the coefficients of the first fundamental form of the surface $f = u^3 = constant$ in variables (u^1, u^2, u^3). Note that in this case, the surface $f = constant$ is parametrized by the surface variables (u^1, u^2) but the function $g'_{(2)}$ is dependent on the variables (u^1, u^2, u^3). The variable u^3 enters this function because it is the value of the constant. In accordance with the notation introduced in [1], the prime on $g'_{(2)}$ means that it is a function of u^3.

Using Eq. (4) on the right side of Eq. (2) gives the following result after integrating with respect to variable u^3:

$$I = -\int \left\{ \frac{\partial}{\partial u^3} \left[\frac{\varphi \sqrt{g'_{(2)}}}{|\nabla f|} \right] \right\}_{u^3 = 0} du^1 du^2 \qquad \text{(EQ 5)}$$

$$= -\int \left\{ \frac{1}{|\nabla f|} \frac{\partial}{\partial n} \left[\frac{\varphi \sqrt{g'_{(2)}}}{|\nabla f|} \right] \right\}_{u^3 = 0} du^1 du^2$$

Taking the normal derivative in the integrand and using the following result from differential geometry [2]

$$\frac{\partial}{\partial n} \sqrt{g'_{(2)}} = -2H_f \sqrt{g'_{(2)}}, \qquad \text{(EQ 6)}$$

we get

$$\int \varphi(\boldsymbol{x}) \delta'(f) d\boldsymbol{x} = \int_{f = 0} \left\{ -\frac{1}{|\nabla f|} \frac{\partial}{\partial n} \left[\frac{\varphi(\boldsymbol{x})}{|\nabla f|} \right] + \frac{2H_f \varphi(\boldsymbol{x})}{|\nabla f|^2} \right\} dS \qquad \text{(EQ 7)}$$

where H_f is the local mean curvature of the surface $f = constant$ in Eq. (6) and of the surface $f = 0$ in Eq. (7). Here we have used the following result for element of the surface area of $f = 0$:

$$dS = \left[\sqrt{g'_{(2)}} \right]_{u^3 = 0} du^1 du^2 = \sqrt{g_{(2)}} du^1 du^2 \qquad \text{(EQ 8)}$$

where $g_{(2)}$ is simply the determinant of the coefficients of the first fundamental form of the surface $f = 0$ in variables (u^1, u^2). Equation (7) is the main result of this paper. It was first explicitly given by the author in [1]. Another way of writing Eq. (7) is

$$\int \varphi(\boldsymbol{x}) \delta'(f) d\boldsymbol{x} = \int_{f = 0} \left\{ -\frac{1}{|\nabla f|^2} \frac{\partial \varphi}{\partial n} + \frac{\varphi(\boldsymbol{x}) \boldsymbol{n} \cdot \nabla |\nabla f|}{|\nabla f|^3} + \frac{2H_f \varphi(\boldsymbol{x})}{|\nabla f|^2} \right\} dS \qquad \text{(EQ 9)}$$

Note that $\boldsymbol{n} \cdot \nabla |\nabla f| = \partial^2 f / \partial n^2$ so that the integral on the right of Eqs. (7) and (9) depend on both $\partial f / \partial n = |\nabla f|$ and $\partial^2 f / \partial n^2$.

2.1 Special Cases

It can be seen that the integral I of Eq. (2) is not invariant under the change of description of the surface $f = 0$. The same surface S defined by two different functions $f_1 = 0$ and $f_2 = 0$ will, in general, give two different values of I. This means that the integrand of this equation, in the form written above, does not usually appear in applications. We will next give an integral involving $\delta'(f)$ which is invariant under the change of description of the surface $f = 0$. In finding this integral, we have been guided by examples similar to Eq. (1). If we assume that the function $\varphi(\boldsymbol{x})$ is given by Eq. (3), then, Eq. (7) becomes

$$\int Q(\boldsymbol{x}, t)|\underset{\sim}{\nabla} f||\nabla f|\delta'(f)d\boldsymbol{x} = \int_{f=0} \left\{ -\frac{\partial Q}{\partial n} + 2H_f Q \right\} dS \qquad \text{(EQ 10)}$$

This is the invariant result that we were seeking as seen from the right side since the mean curvature is a quantity that does not depend on the representation of the surface $f = 0$. In applications, we do get the function $|\underset{\sim}{\nabla} f||\nabla f|$ always multiplying $\delta'(f)$. For example, in Eq. (1), by assumption $|\underset{\sim}{\nabla} f| = 1$ and we see that $|\underset{\sim}{\nabla} f||\nabla f| = |\nabla f|$ does multiply $\delta'(f)$.

In case we have defined $|\underset{\sim}{\nabla} f| = 1$, i.e., $|\nabla f| = 1$ on $f = 0$, then Eq. (10) becomes

$$\int Q(\boldsymbol{x}, t)|\nabla f|\delta'(f)d\boldsymbol{x} = \int_{f=0} \left\{ -\frac{\partial Q}{\partial n} + 2H_f Q \right\} dS \qquad \text{(EQ 11)}$$

This is a very useful result because the assumption of $|\underset{\sim}{\nabla} f| = 1$ can considerably simplify algebraic manipulations involving generalized functions [1]. The readers may find the applications of this equation to wave propagation problems in [1, 2].

Note that in some of the publications of the author, the function $|\nabla f|$ was left out of the integrand on the left side of Eq. (11) [1,2]. However, since this function was also left out in the manipulation of the source terms of a wave equation, mistaken for $|\underset{\sim}{f}|$ which was assumed equal to 1, the correct final results were obtained. The author is deeply indebted to Dr. Alexandre I. Saichev of the Radiophysical Department of Nizhni Novgorod University in Russia, for bringing this mistake to his attention [3].

3.0 References

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